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# Accelerated Kaczmarz for Convergence Speed-up in Multi-Contrast Magnetic Particle Imaging

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**Abstract**

Magnetic Particle Imaging (MPI) is a tracer based medical imaging modality with great potential due to its high sensitivity, high spatio-temporal resolution, and ability to quantify the tracer concentration. Image reconstruction in MPI is an ill-posed problem, which can be addressed by the use of regularization methods. The corresponding optimization problem is most commonly solved using the Kaczmarz algorithm. Reconstruction using the Kaczmarz method for single-contrast MPI is very efficient as it produces the desired images fast with a small number of iterations. For multi-contrast MPI, however, the regular Kaczmarz algorithm fails to obtain good quality images without channel leakage when using a small number of iterations. In this work, we propose to use an accelerated Kaczmarz method in order to reduce the reconstruction time needed to achieve a good separation of the channels and a good image quality in multi-contrast MPI.

## I. Introduction

Magnetic particle imaging (MPI) is an emerging medical imaging technique that enables sensitive and fast imaging of magnetic nanoparticles [1]. Beyond known applications such as the diagnosis of cardiovascular diseases, the field of multi-contrast MPI has opened the door for many applications. Multi-contrast MPI enables separate reconstruction of the signal from different tracer materials or environments, which results in multi-channel images presenting different tracer or environment properties, such as, temperature [2], viscosity [3], mobility state, material [4], or core size [5]. For instance, scientists think of it as a safe alternative for tracking of devices in interventional imaging since it does not require ionizing radiation [6]. In this particular case, different types of magnetic particles are used to visualize the catheter and the vessel in separate channels.

In MPI, the reconstruction problem is usually solved using the Kaczmarz method [7] as it yields the desired results efficiently with a small number of iterations. However, when dealing with the reconstruction of multi-contrast MPI data, one faces the problem that the Kaczmarz algorithm requires a high number of iterations in order to obtain a good separation of the channels. For instance in [3, 5], up to 10.000 iterations were required. To address this issue, this work proposes the application of an accelerated Kaczmarz algorithm [8] to improve the convergence speed of multi-contrast MPI reconstruction.

# II. Methods and materials

## II.I. Accelerated Kaczmarz Method

The main idea of the acceleration method is to transform the sequence of vectors generated by Kaczmarz' method $\boldsymbol{x}_n$ into another sequence $\boldsymbol{y}_k^n$ which converges faster to the same limit using a sequence transformation technique. Acceleration of the Kaczmarz algorithm can be obtained in two different ways, firstly by directly applying the sequence transformation to the sequence of Kaczmarz vectors, and secondly by restarting the Kaczmarz algorithm at each iteration with the transformed vector. The acceleration also depends on the choice of the sequence transformation technique. In this abstract, for the sake of simplicity, we only show results obtained by applying the restarted accelerated Kaczmarz method with the Reduced Rank Extrapolation (RRE) transformation technique [8].

The sequence transformation $(\boldsymbol{x}_n) \to (\boldsymbol{y}_k^n)$ is defined as

$$\boldsymbol{y}_k^n = \boldsymbol{x}_n - \alpha_1^n \Delta \boldsymbol{x}_n - \ldots - \alpha_k^n \Delta \boldsymbol{x}_{n+k-1}, \quad k = 0, 1, \ldots \tag{1}$$

where the parameters $\alpha_i^n$ are the solutions of the linear system

$$\delta d_{i,0}^{(n)} \alpha_1^n + \ldots + \delta d_{i,k-1}^{(n)} \alpha_k^n = d_{i,0}^{(n)}, \; i = 1, \ldots, k,$$

where $\delta$ is the difference operator defined by $\delta d_{i,j}^{(n)} = d_{i,j+1}^{(n)} - d_{i,j}^{(n)}$ and $k$ is the parameter that decides the number of Kaczmarz vectors required to perform the sequence transformation. The $d_{i,j}^{(n)}$ is defined according to RRE as

$$d_{i,j}^{(n)} = \langle \Delta^2 \boldsymbol{x}_{n+i-1}, \Delta \boldsymbol{x}_{n+j} \rangle,$$

where $\Delta$ is also a difference operator defined by $\Delta \boldsymbol{x}_n = \boldsymbol{x}_{n+1} - \boldsymbol{x}_n$ and $\Delta^2 \boldsymbol{x}_n = \boldsymbol{x}_{n+1} - 2\boldsymbol{x}_n + \boldsymbol{x}_{n-1}$.

Algorithm 1 shows the pseudo code of the restarted Kaczmarz method, taking into account the MPI forward model $\boldsymbol{A}\boldsymbol{x} + \boldsymbol{n} = \boldsymbol{b}$, where $\boldsymbol{A}$ is the system matrix, $\boldsymbol{b}$ is the measurement vector, $\boldsymbol{x}$ is the particle concentration and $\boldsymbol{n}$ is the noise vector.

**Algorithm 1** Restarted Accelerated Kaczmarz (RK)

**Input:** $\boldsymbol{A} \in \mathbb{R}^{M\times N}, \boldsymbol{b} \in \mathbb{R}^M, \boldsymbol{x}_0 \in \mathbb{R}_+^N$
**Choose:** $k \in \mathbb{N}, k \geq 1, \lambda \geq 0$
**Set:** $l = k + 1$
**Set:** $\boldsymbol{v}^0 = 0$
  **for** $n = 0, 1, \ldots, N$ **do**
    **for** $j = 0, \ldots, l-1$ **do**
      $\boldsymbol{p}_0 \leftarrow \boldsymbol{x}_j$
      **for** $i = 1, \ldots, M$ **do**
        $\beta_i \leftarrow \frac{b_i - \langle \boldsymbol{p}_{i-1}, \boldsymbol{a}_i \rangle - \lambda^{\frac{1}{2}} v_i}{\|\boldsymbol{a}_i\|_2^2 + \lambda}$, $\boldsymbol{a}_i$ is the i-th row of $\boldsymbol{A}$
        $\boldsymbol{p}_i \leftarrow \boldsymbol{p}_{i-1} + \beta_i \boldsymbol{a}_i$
        $v_n^i \leftarrow v_n^{i-1} + \beta_i (\lambda)^{\frac{1}{2}}$
      **end for**
      $\boldsymbol{x}_{j+1} \leftarrow \boldsymbol{p}_M$
    **end for**
    Compute $\boldsymbol{y}_k^{(0)}$ by equation (1)
    $\boldsymbol{z}_n \leftarrow \boldsymbol{y}_k^{(0)}$
    $\boldsymbol{x}_0 \leftarrow \boldsymbol{z}_n$
  **end for**

## II.II. Simulation

To evaluate the accelerated Kaczmarz, a 2D simulation study is performed using two system matrices, both measured using a preclinical MPI system (Bruker, Ettlingen, Germany). The system matrices were measured on a $30 \times 30$ grid with drive field amplitudes of 12 $\mathrm{mT}\mu_0^{-1}$ in x- and y- directions and a gradient strength of $G_x = G_y = -1\ \mathrm{Tm}^{-1}\mu_0^{-1}$ and $G_z = 2\ \mathrm{Tm}^{-1}\mu_0^{-1}$. The delta sample had a size of $1 \times 1 \times 1\mathrm{mm}^3$. Both samples were filled with MPI tracer perimag (micromod Partikeltechnologie GmbH, Rostock, Germany); one with mobilized (liquid) perimag and the other with immobilized (solid) perimag. As an examplary imaging experiment we consider the scenario of catheter tracking in a stenosis. The following forward model is used to create the simulated data

$$\underbrace{\begin{pmatrix} \boldsymbol{A}_1 & \boldsymbol{A}_2 \end{pmatrix}}_{\boldsymbol{A}} \underbrace{\begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{pmatrix}}_{\boldsymbol{x}} + \boldsymbol{n} = \boldsymbol{b},$$

where $\boldsymbol{A}$ is the system matrix that consists of the matrices acquired for the two channels concatenated along the spatial axis, $\boldsymbol{n}$ is the white Gaussian noise vector and $\boldsymbol{b}$ is the measurement vector. The aim is to find the concentration vector $\boldsymbol{x}$ which includes the signal of the two channels $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. The original phantom used in the simulation is shown in Figure 2.

# III. Results

The acceleration offered by the restarted Kaczmarz algorithm depends on the choice of $k$. Figure 1 shows the NRMSD of the reconstructions done with regular Kaczmarz method and the restarted Kaczmarz for different $k$.

 

The figure shows that the higher $k$ is set, the faster the NRMSD is reduced, specially for the second channel.

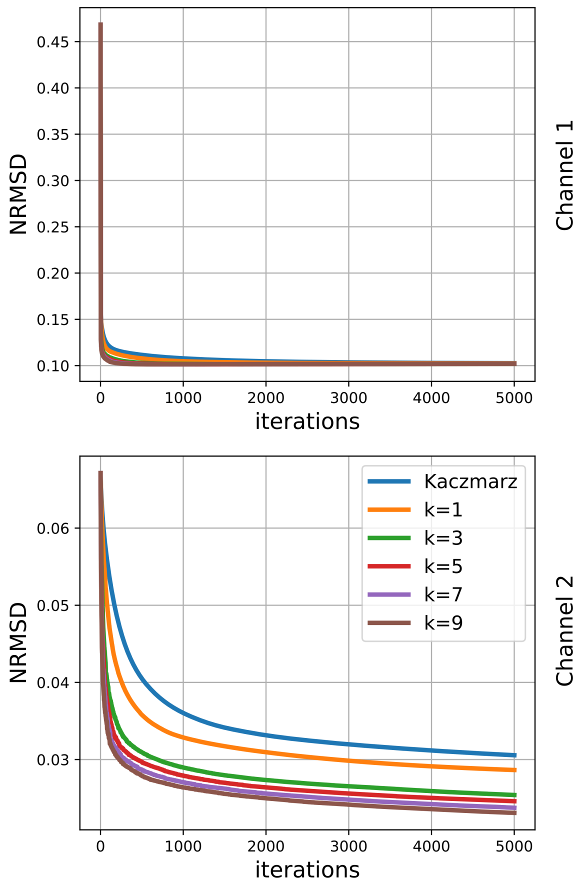


**Figure 1:** The NRMSD is shown for the reconstructions of the first and second channels of the catheter tracking experiment, with channel 1 showing the stenosis and channel 2 showing the catheter tip, using the regular Kaczmarz and the restarted Kaczmarz for several $k$. The $x$-axis represents the outer number of iterations $N$.

Figure 2 shows the reconstruction results obtained with the regular and restarted Kaczmarz methods with $k = 1$ and 5 for different number of iterations. The improvement of the restarted Kaczmarz method is clearly visible even for 10 iterations, as seen in the first row. For the sake of evaluating the proposed acceleration method, an NRMSD threshold is defined to quantify the required number of iterations. The NRMSD threshold is set to $\text{Thresh}_{\text{NRMSD}} = 0.0305$, which offers an almost complete separation of the channels. The number of required iterations and the NRMSD threshold are applied on the second channel as it suffers more from the channel leakage. The number of iterations needed to obtain a good quality images satisfying the NRMSD threshold and complete separation of the channels is with 5000 the highest using regular Kaczmarz method. Using the restarted Kaczmarz method with $k = 1$ reduces the needed number of iterations to 2310 and with $k = 5$, the number of iterations is reduced even further to 370. The runtime of the whole reconstruction scheme is also reduced by a factor of 1.1 for $k = 1$ and by a factor of 2.5 for $k = 5$.

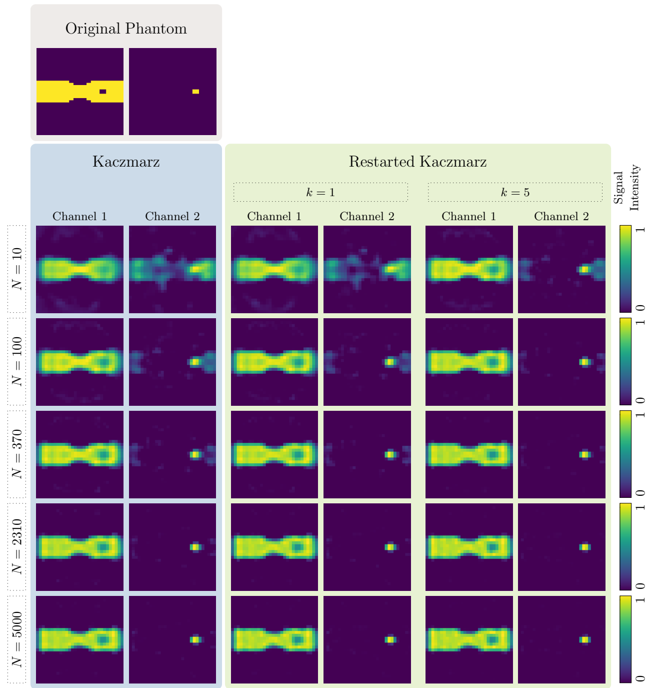


**Figure 2:** The reconstruction results are shown for the regular Kaczmarz and the restarted Kaczmarz with $k = 1$ and 5 for different numbers of iterations.

# IV. Discussion

The reconstruction using the Kaczmarz method in MPI usually requires 3 to 10 iterations to obtain good quality reconstructions, but that is not the case for multi-contrast MPI because of the channel leakage. The first two images in the first row of Figure 2 show the amount of signal leakage from channel 1 into channel 2 when using Kaczmarz algorithm with 10 iterations. Figures 1 and 2 show that the effectiveness of our acceleration method increases with $k$. However, one cannot choose $k$ very high, as the acceleration technique becomes computationally more demanding and requires more storage. Choosing $k$ is quite challenging as there is a trade off between the reduction of the number of Kaczmarz iterations and the computational costs and the required storage per each iteration. Mostly, picking $k$ up to 5 yields a good trade-off between memory demand and acceleration.

# V. Conclusion

The proposed accelerated Kaczmarz method managed to reduce the number of Kaczmarz iterations required to obtain good quality reconstruction and good channel separation for the catheter-stenosis simulation we considered. The number of needed Kaczmarz iterations is decreased by a factor of 2.15 for $k = 1$ and by a factor of 13.5 for $k = 5$. The reconstruction time is also reduced using this acceleration method by a factor dependent on the choice of $k$, as the choice of $k$ decides the number of inner iterations conducted. The accelerated Kaczmarz method will be a good tool for achieving faster MPI reconstructions, particularly for multi-contrast applications but also for cases where a sparsity prior [9] is applied. Comparing both the accelerated and the restarted accelerated Kaczmarz methods for various sequence transformation techniques will be considered for future work.

## Author's statement

Conflict of interest: Authors state no conflict of interest.

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