International Journal on Magnetic Particle Imaging
Vol 10, No 1, Suppl 1, Article ID 2403006, 4 Pages

Infinite Science Publishing

Proceedings Article

Ellipsoidal harmonic expansions for efficient approximation of magnetic fields in medical imaging

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Abstract

Exact knowledge on the magnetic fields used in tomographic imaging systems is important for accurate sequence planning and model-based image reconstruction in existing devices. It is common to measure the magnetic fields on a small subset of points and use spherical harmonics to approximate the fields inside a spherical region. However, most scanner bores could be better filled using a cylindrical or ellipsoidal region. In this work, we present the application of ellipsoidal harmonics for efficient approximation of a typical magnetic field in magnetic particle imaging.

I. Introduction

In tomographic imaging methods based on magnetic fields, e.g. magnetic resonance imaging and magnetic particle imaging (MPI), the exact values of the magnetic fields in the field of view (FOV) are of great interest and play a major role in signal encoding [1, 2]. Depending on the scanners underlying coil setup, the magnetic fields are by no means ideal in the FOV. For instance in MPI, knowledge on the non-linearity of the magnetic fields can be used for accurate sequence design, to speed up the calibration process when using multiple patches [3], or for higher accuracy of model-based system matrices [4]. Furthermore, knowing the exact values of the magnetic fields is important for the development and verification of new receive coils or field generators [5, 6].

The magnetic fields can be measured using a magnetic field sensor attached to a robot. The naive ansatz, covering every position on a highly resolved three-dimensional grid would be a long and tedious process. For this reason, more efficient measurement methods were introduced in which the field can be expanded very efficiently into a series of spherical harmonic functions with only a few measurement points on a spherical surface in the form of t-designs [7, 8]. However, typical scanner bores have a cylindrical shape, such that several spheres, shifted along the bore axis, have to be measured to cover the entire FOV. But even when the spheres are shifted with big overlap, this procedure leaves areas uncovered by any of the spheres. In this work we instead consider an ellipsoidal volume, which much better covers typical scanner bores. To this end, we consider ellipsoidal harmonic functions and the associated series expansion, which has not yet been applied to describe magnetic fields in tomographic imaging so far. However, in astrophysics calculations of the gravitational potential of (non ideally round) celestial bodies get more efficient using ellipsoidal harmonics [9]. In this study we show,

how ellipsoidal harmonics can be used to efficiently measure and represent magnetic fields in MPI.

II. Methods and Materials

II.I. Theory

Magnetic fields in a source-free region $\Omega_{\text{FOV}} \subset \mathbb{R}^3$ can be assumed to be harmonic, i.e. they fulfill Laplace's equation. This applies to all magnetic fields $\boldsymbol{B}:\mathbb{R}^3 \to \mathbb{R}^3$ that are located within the bore of an MPI scanner, as no currents flow there. The transformation of the Laplace equation $\Delta B_i(\boldsymbol{x}) = 0$, for $i \in \{1,2,3\}$ and $\boldsymbol{x} \in \Omega_{\text{FOV}}$ into an ellipsoidal coordinates, starts with the definition of a reference ellipsoid

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} = 1, \tag{1}$$

where $0 < a_3 < a_2 < a_1$ are the three semi-axes and $x_1, x_2, x_3 \in \mathbb{R}$ Cartesian coordinates. The linear eccentricities of the reference ellipsoid are given as

$$h_1 = \sqrt{(a_2^2 - a_3^2)},\, h_2 = \sqrt{(a_1^2 - a_3^2)},\, h_3 = \sqrt{(a_1^2 - a_2^2)}. \tag{2}$$

By introducing an orthogonal ellipsoidal coordinate system ρ, μ, ν with $0 \le \nu^2 \le h_3^2 \le \mu^2 \le h_2^2 \le \rho^2 < \infty$ and performing a separation of variables on the Laplace equation, one ends up with the so-called Lamé equation [10], whose solutions $E_n^m : \mathbb{R} \to \mathbb{R}$ for $m \in \mathbb{N}, n \in \mathbb{N}_0$ are called Lamé functions. Lamé functions can be divided in four classes

$$\begin{aligned}
\mathcal{K} &= \{P(x)\},\\
\mathcal{L} &= \{\sqrt{|x^2 - h_3^2|}P(x)\},\\
\mathcal{M} &= \{\sqrt{|x^2 - h_2^2|}P(x)\},\\
\mathcal{N} &= \{\sqrt{|x^2 - h_3^2|}\sqrt{|x^2 - h_2^2|}P(x)\},
\end{aligned}$$

where $P(x) = \alpha_0 x^n + \alpha_1 x^{n-2} + ... + \alpha_k x^{n-2k} + ...$ is a polynomial of maximum degree n and $x \in \{\rho, \mu, \nu\}$. There are always $2n+1$ Lamé functions of degree n, distributed over the four classes and sorted using the index $m = 1, ..., 2n+1$. The conversion between Cartesian coordinates (x_1, x_2, x_3) and ellipsoidal coordinates (ρ, μ, ν) is not straightforward and additional sign conventions have to be respected to achieve uniqueness. We refer the interested reader to [11]. Lamé functions of each class can be determined for a given reference ellipsoid, by solving an $\mathcal{O}(n^3)$ eigenvalue problem for each class and each $n \in \mathbb{N}$ [10], leading to the inner ellipsoidal harmonics $\mathbb{E}_n^m(\rho,\mu,\nu) = E_n^m(\rho)E_n^m(\mu)E_n^m(\nu)$ and the surface ellipsoidal harmonics $S_n^m(\mu,\nu) = E_n^m(\mu)E_n^m(\nu)$. On the ellipsoidal surface holds $\rho = a_1$. With help of the inner ellipsoidal harmonics $\mathbb{E}_n^m$ with $n \le L$, which pose a basis of the harmonic polynomials with maximum degree $L \in \mathbb{N}$, the magnetic field components can be calculated inside the reference ellipsoid (by placing Dirichlet boundary conditions on the surface) using the following expansion:

$$B_i(\rho,\mu,\nu) \approx \sum_{n=0}^{L} \sum_{m=1}^{2n+1} A_n^m \mathbb{E}_n^m(\rho,\mu,\nu), \tag{3}$$

where

$$A_n^m = \frac{1}{\gamma_n^m E_n^m(a_1)} \oint_{S_{a_1}} B_i(a_1,\mu,\nu) S_n^m(\mu,\nu)\, \mathrm{d}\Omega(\mu,\nu), \tag{4}$$

$$\gamma_n^m = \oint_{S_{a_1}} S_n^m(\mu,\nu)^2\, \mathrm{d}\Omega(\mu,\nu), \tag{5}$$

where S_{a_1} is the surface of the reference ellipsoid. Equation (3) is exact, if the magnetic field can be described by a polynomial of maximum degree L. Under this assumption, the surface integrals in (4) can be calculated exactly using a finite sum over an ellipsoidal t-design with $t \ge 2L$ (this follows from the existence of a transformation between ellipsoidal surface integrals and spherical surface integrals given in [10]). To this end, the Cartesian coordinates (y_1, y_2, y_3) of a spherical t-design need to be shifted to the ellipsoid surface using

$$x_1 = y_3 a_1,\ x_2 = y_1\sqrt{a_1^2 - h_3^2},\ x_3 = y_2\sqrt{a_1^2 - h_2^2},$$

and converted to ellipsoidal coordinates afterwards.

II.II. Simulations

To demonstrate the accuracy and applicability of the presented expansion, we consider the x-drive field of the preclinical MPI system (25/20 FF) from Bruker Corporation, Ettlingen, Germany. Field amplitude values calculated by the FEM software COMSOL Multiphysics[1] are used on the ellipsoidal surface points given by the t-design. To highlight the benefits of ellipsoidal harmonics, we want the magnetic field to be calculated inside a cylinder of 21.2 cm length and 9 cm diameter, which are the dimensions of the dedicated receive coil presented in [5]. To this end, the reference ellipsoid is set using the semi-axis $\boldsymbol{a} = (17.5, 5.61, 5.6)$ cm and an ellipsoidal 12-design with 86 points is chosen, as displayed in Figure 1. The implementation of the ellipsoidal harmonic expansion was written in Julia.

III. Results

The calculated field for $z = 0$ on an 71×23 rectangular grid using the ellipsoidal harmonic expansion is shown

[1]COMSOL Multiphysics v.6.0. www.comsol.com. COMSOL AB, Stockholm, Sweden

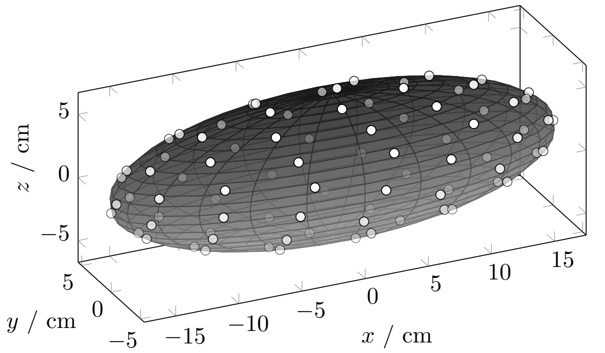

Figure 1: Surface-plot of the reference ellipsoid with semi-axis $\boldsymbol{a} = (17.5, 5.61, 5.6)$ cm. The 86 points of an ellipsoidal 12-design are displayed on top.

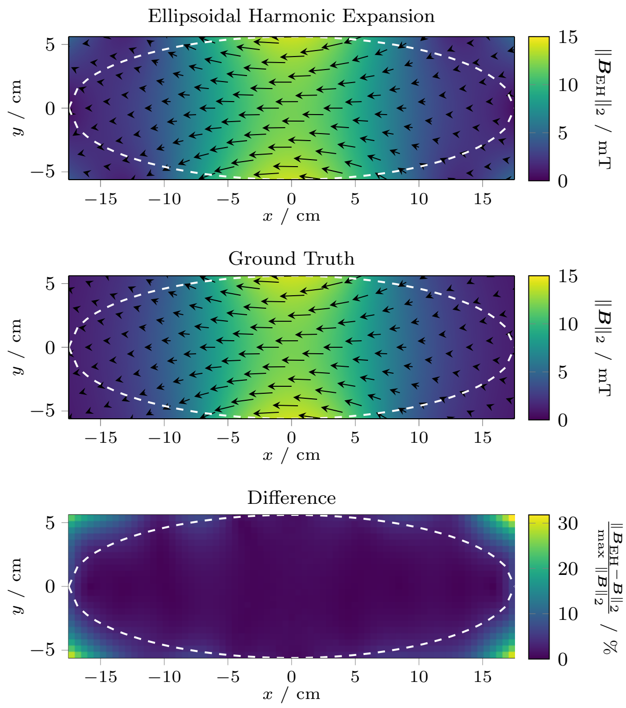

Figure 2: Shown are the calculated magnetic fields in the xy-plane for $z = 0$ using ellipsoidal harmonics (top) and the simulation software COMSOL (middle). The norm is color-coded, the field direction is displayed using field arrows. Furthermore, a difference plot is given (bottom).

in the upper plot in Figure 2. The magnetic field aligns in negative x-direction, as it would be expected for the x-drive field. For comparison, the simulated magnetic field using COMSOL is shown on the same grid as well as a difference plot. The absolute error inside the ellipsoid has a mean value of 0.1 mT equaling 1.5 % of the mean absolute value of the simulated field and gets slightly larger towards the edges of the ellipsoid (with a maximum absolute error of 0.8 mT). Outside the ellipsoid, the absolute error gets very large, which can be even observed in the plotted fields and not only in the difference plot.

IV. Discussion and Conclusion

With the presented ellipsoidal harmonic expansion, a typical MPI drive-field could be represented effectively using only 86 data points for the above given reference ellipsoid enveloping a cylinder of 21.2 cm length and 9 cm diameter. Using spherical harmonics to represent the magnetic field in the full cylinder, at least three separate spheres along the x-axis would be necessary.

The small absolute error of the field inside the ellipsoid underlines the accuracy of the ellipsoidal expansion. It was to be expected, that the error raises outside the ellipsoid, since equation (3) is not valid there. The slightly increasing error close to the edge of the ellipsoid could be circumvented by using a reference ellipsoid, which is slightly bigger than the FOV. Moreover, a lower truncation error could be expected, when using a larger ellipsoidal t-design, with $t > 12$. When using the method on measured data, the rather small approximation error has to be contextualized to the displacement error of the robot and the accuracy of the magnetic field sensor.

Concluding, ellipsoidal harmonic expansion proves to be a very powerful tool for the effective measurement and representation of magnetic fields in tomographic imaging devices. The effectiveness of this method was demonstrated through the simulation of a drive-field on a preclinical MPI-scanner. In comparison to the currently used spherical harmonics, two more degrees of freedom allow to much better match the region of interest. This is especially useful for an oval scanner bore [12]. An in-dept error analysis as well as a profound variety of test cases remain necessary for future work.

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